Written exam on Lab-on-a-Chip course, Spring Semester 2009
June 8th, 2009

Examination time: 4 hours (9am – 1pm)
Allowed means: MYO "Fundamentals of Fluid Mechanics", lecture slides and a calculator (no computer or PDAs, please)
Complete solution should include all equations calculated down to a numerical answer. Numerical answer alone is not counted as a solution.
Some useful constants for your problems are listed in the end of the exam paper

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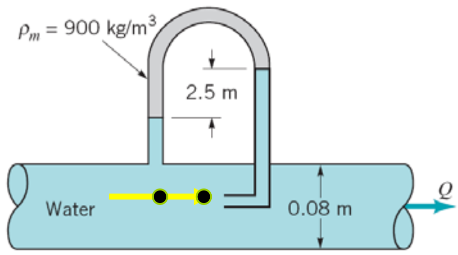
Problem 1. Determine flow rate through the pipe shown in the figure.



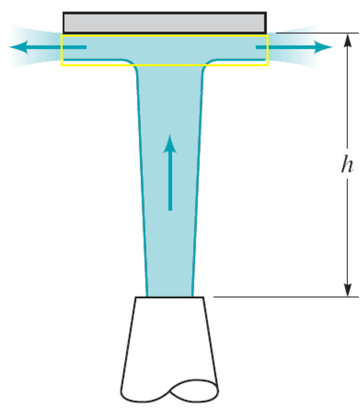
Solution:
The manometer shows the difference between static and stagnation pressure at points (1) and (2) along the same streamline. According to the manometer:

$$p_1 - x\rho_{water}\cdot g - 2.5\cdot\rho_m\cdot g + 2.5\cdot\rho_{water}\cdot g + x\rho_{water}\cdot g = p_2$$

$$p_2 - p_1 = 2.5g\cdot(\rho_{water} - \rho_m) = \frac{\rho_{water}v^2}{2}$$

$$v = 2.2\frac{m}{s};\quad Q = A\cdot v = 0.011\frac{m^3}{s}$$

Problem 2. A vertical jet of water leaves a nozzle at a speed of 10 m/s and a diameter of 20mm. It suspends a plate of 1.5 kg. Find the distance h.



Solution:

Let's choose control volume as shown in the figure. Reynolds transport theorem applied to the momentum:

$$\rho v_1(-v_1)\cdot\pi r_1^2 = -1.5\cdot g$$

(we neglect the weight of water)
Velocity at the entrance into the volume can be determined from

Bernoulli equation and the radius from the continuity equation:

$$\frac{\rho v_0^{\ 2}}{2} = \frac{\rho v_1^{\ 2}}{2} + \rho g h$$

$$v_0 \pi r_0^{\ 2} = v_1 \pi r_1^{\ 2}$$

So,

$$v_1 = \frac{1.5g}{\rho v_0 \pi r_0^{\ 2}} = 4.7 \tfrac{m}{s}$$

$$h = \frac{1}{2g}\left(v_0^{\ 2} - v_1^{\ 2}\right) \approx 4m$$

Problem 3. The velocity components of a 2D velocity field are given by the equations:

$$u = y^2 - x(1+x)$$

$$v = y(2x+1)$$

(a) Find the stream function
(b) Find the acceleration
(c) Show that the flow is irrotational and satisfies the conservation of mass

<u>Solution</u>

(a) $\frac{y^3}{3} - y(x+x^2) + const$

(b) local acceleration is zero (no time dependence), convective acceleration:

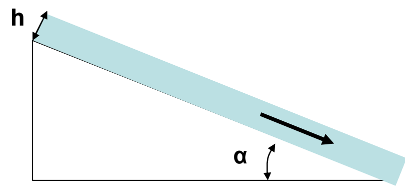
$$a_x = \left[y^2 - x(1+x)\right]\cdot(-1-2x) + y(2x+1)\cdot 2y$$

$$a_y = \left[y^2 - x(1+x)\right]\cdot(2y) + y(2x+1)\cdot(2x+1)$$

(c)

$$\text{curl}\vec{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2y - 2y = 0$$

$$\text{div}\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -1 - 2x + 2x + 1 = 0$$



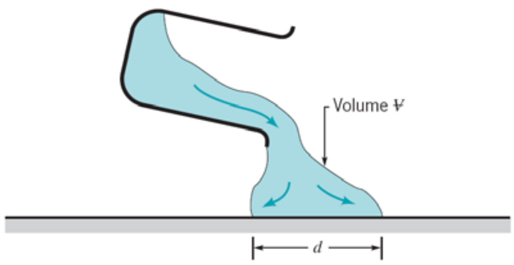
Problem 4. Using Navier-Stokes equation, determine relation between the volumetric flow rate and the height **h** of the layer of viscous liquid of constant thickness flowing steadily down an infinite inclined plane (inclination angle **α**). Assume laminar flow and negligible air resistance.

Solution:
Taking into account no pressure gradient we can solve Navier-Stokes equation with boundary conditions: "no slip" at the lower surface and zero stress at the free surface.

$$u = \frac{\rho g}{\mu}\sin\alpha\left(hy - \frac{y^2}{2}\right)$$

Integrating from 0 till h and multiplying by the width, we get the volume flow rate per unit width

$$Q = \frac{\rho g h^3}{3\mu}\sin\alpha$$



Problem 5. A viscous liquid is poured onto a horizontal plate as shown in the figure. Assume the time ***t*** required for the fluid to flow a certain distance ***d*** along the plate is a function of a volume poured ***V***, acceleration of gravity ***g***, fluid density ρ and viscosity ***μ***. Determine the set of Pi-terms to describe the process.

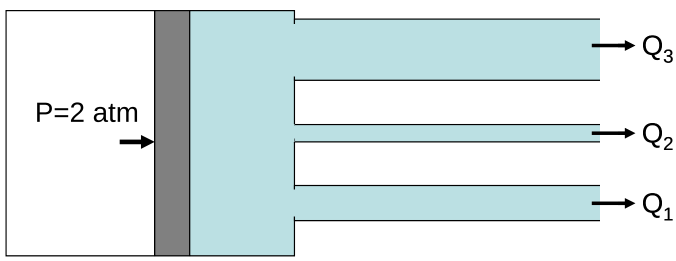
Solution:

$$t = \phi(d, V, g, \rho, \mu)$$

We have 6 variables and 3 dimensions, the number of Pi-terms 6-3=3
Pi terms could be:

$$t\sqrt{\frac{g}{d}};\frac{V}{d^3};\frac{\rho}{\mu}\sqrt{gd^3}$$

$$t\sqrt{\frac{g}{d}} = \phi\left(\frac{V}{d^3},\frac{\rho}{\mu}\sqrt{gd^3}\right)$$



Problem 6. Commonly in "industrial" microfluidics, the liquid is propelled by air pressure that can be switched On and Off with a simple valve. In a circuit shown in the figure, the liquid flows through three rectangular channels 50μm, 100 μm and 200 μm wide. The height of the channels is equal to 50 μm and the length is 1 cm. Calculate the volumetric flow rate through each channel at a pressure of 2 atm. What will be the Re numbers for each channel at this condition?

Solution:
Hydraulic diameter of a non/circular channel can be calculated as:

$$D_h = \frac{4A}{P} = \frac{4 \times width \times height}{2 \times (width + height)}$$

For the channels in the problem it will be, 50um, 67 um and 80 um, correspondingly
The pressure drop can be expressed using friction factor commonly given as C=f·Re:

$$\Delta p = f \frac{l}{D_h} \frac{\rho V^2}{2} = (f \,\mathrm{Re}) \frac{V \mu l}{2 D_h^{\,2}} = (f \,\mathrm{Re}) \frac{Q \mu l}{2 D_h^{\,2} A}$$

For square channel 50x50 um C=56.9, for 50x100um C=62.2, for 50x200um C=72.9 (see MYO, p.448)

$$V = \frac{2 \Delta p D_h^{\,2}}{\mu l (f \,\mathrm{Re})} = 1.78; 2.90 \text{ and } 3.56 \text{ m/s}, \text{ correspondingly.}$$

Flow $Q = height \times width \times V = 4.45 \cdot 10^{-9} \frac{m^3}{s}; 1.45 \cdot 10^{-8} \frac{m^3}{s}; 3.56 \cdot 10^{-8} \frac{m^3}{s}$, correspondingly.

$$\mathrm{Re} = \frac{V D_h \rho}{\mu} = 89; 193; 285., \text{ correspondingly.}$$

The flow is a laminar one. Our use of the equations for laminar flow was justified.

Problem 7. Electroosmotic effect is used to propel 10mM NaCl solution through a channel that is 200um wide, 50um high and 20mm long. What voltage should be applied between the inlet and the outlet to achieve flow through the channel equal to 1ul/min. Use approximation of flow between two infinite parallel plates, zeta potential on the wall is -100mV

Solution
Debye length for 10mM solution is expected to be small (around 3nm) and the flow can be considered as a plug flow. The velocity will be:

$$u = -\frac{\varepsilon \varsigma}{\mu} E = -\frac{78 \varepsilon_0 \varsigma}{\mu} \frac{V}{l}; \;\; V = \mu l \frac{u}{78 \varepsilon_0 \varsigma}$$

Linear velocity can be found dividing the flow rate by the cross section:

$u = 1.7e-3$, so we need $V \approx 500V$

List of constants:
Density of water 1000 kg/m^3;

Viscosity of water $1 \cdot 10^{-3}$ Pa*s
Permittivity of vacuum: $\varepsilon_0 = 8.854 \cdot 10^{-12}$ C/ (N·m^2).
Relative permittivity of water $\varepsilon = 78$